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Infinities of the massless spin-$\frac{1}{2}$ triangle diagram in quantum gravity

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Abstract. In a massless fermion–graviton interaction field theory, the triangle graph with three external graviton lines is considered. The infinities arising from this and associated graphs are calculated in the dimensional regularization scheme, putting the energy–momentum of one of the external legs equal to zero. These infinities are exactly cancelled by those of the three-graviton graph originating from the graviton self-energy counter-Lagrangian density.

1. Introduction

Recently, attention has been paid to the calculation of various graviton self-energy contributions and to the counter-Lagrangians needed to cancel the infinities arising. Dimensional regularization manifested itself thereby as a useful tool (Capper 1973, Capper and Duff 1973, Capper *et al* 1973a, b and De Meyer 1974, to be referred to as I). Higher-order diagrams in quantum gravity were also briefly considered (Capper and Leibbrandt 1974).

In the present paper, we will consider the problem of a spin-$\frac{1}{2}$ particle triangle diagram with graviton lines attached; for the sake of convenience, we will restrict ourselves to a massless fermion loop-particle. It will be checked whether the infinities arising from the entity consisting of this diagram and two other associated diagrams of the same type, are cancelled, within the dimensional regularization scheme, by the infinities emerging from a three-graviton diagram. The latter diagram originates from the counter term which has resulted from the electron one-loop graviton self-energy calculation in I.

We will specifically choose the weight of the fermion field in such a way that the vertex functions needed appear in their most simplified form for calculation. As a consequence of Borchers' theorem, this simplification is justified in this case. As a second simplification, we will put the energy–momentum of one of the external graviton lines equal to zero. Although complete generality cannot be achieved by this procedure, the proof of our statement is by no means trivial.

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2. The triangle diagram

The interaction of massive spin-$\frac{1}{2}$ particles and gravitons is described by the Lagrangian density of I:

$$\mathscr{L} = \mathscr{L}_{\text{grav}} - \tfrac{1}{2}(-g)^{\Lambda} L^{n\mu}\bar{\psi}\gamma_n \overleftrightarrow{\partial}_\mu \psi - (-g)^{\Lambda}\bar{\psi}(m - \tfrac{1}{4}\epsilon^{klmn} B_{klm}\gamma_n\gamma_5)\psi, \tag{1}$$

where $\mathscr{L}_{\text{grav}}$ is the usual Einstein–Lagrangian density. In (1) we have put:

$$w + \tfrac{1}{2} = \Lambda, \tag{2}$$

the parameter w denoting the weight of the fermion field. The metric used is the Minkowski metric $(+, +, +, -)$, $L^{a\mu}$ is the '*vierbein*' gravity field and B_{klm} is related to the '*vierbein*' connection $B_{\mu lm}$ by:

$$B_{klm} = L^\mu{}_k B_{\mu lm} = L^\mu{}_k L^\nu{}_l (L_{m\lambda}\Gamma^\lambda_{\mu\nu} - \partial_\mu L_{m\nu}). \tag{3}$$

Finally, g stands for the determinant of $g_{\mu\nu}$ and ϵ^{klmn} is the complete antisymmetrical tensor, whereas all other symbols have their usual meaning. In order to obtain the massless fermion theory, the mass is put equal to zero, both in the Lagrangian density and in the Lagrangian counter term. This operation is permitted as long as finite contributions are not taken into account (Capper 1974).

The energy–momentum of one definite external graviton line $g_{\xi\eta}$ (see figure 1) is set equal to zero throughout the paper. The weight of the fermion field is chosen such as to bring the two-fermion–graviton vertex function in a simple form for calculation, eg:

$$V^3_{\alpha\beta}(k; p_1, p_2) = -\tfrac{1}{4}\mathrm{i}\gamma_{(\alpha}q_{\beta)}, \tag{4}$$

with:

$$\gamma_{(\alpha}q_{\beta)} = \tfrac{1}{2}(\gamma_\alpha q_\beta + \gamma_\beta q_\alpha), \tag{5}$$

where q is the difference between the inflowing momenta p_1 and p_2 of the ingoing and outgoing fermion respectively.

In accordance with the dimensional regularization procedure, the energy–momentum vector of the loop particle becomes a vector in 2ω-dimensional Euclidean space. As we are not concerned with the finite contribution from the diagram, it is unnecessary to work in Minkowski space, or to rotate back to Minkowski space at the end of the calculation. Putting

$$-k_1 = +k_2 = k, \tag{6}$$

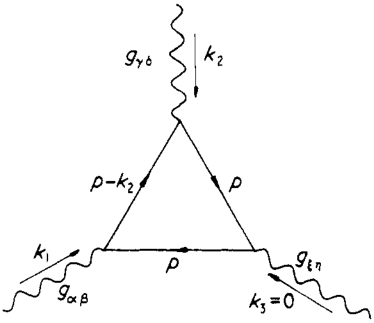

Figure 1. The triangle diagram.

which follows from the energy–momentum conservation law, the amplitude for the triangle diagram can be written as:

$$T_{\alpha\beta,\gamma\delta,\xi\eta}(k) = (-1)\kappa^3 \operatorname{Tr} \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \left(\frac{-i\not{p}}{p^2}\right) \left(-\frac{i}{4}\gamma_{(\xi}2p_{\eta)}\right)$$
$$\times \left(\frac{-i\not{p}}{p^2}\right) \left(-\frac{i}{4}\gamma_{(\alpha}(2p-k)_{\beta)}\right) \left(-i\frac{\not{p}-\not{k}}{(p-k)^2}\right) \left(-\frac{i}{4}\gamma_{(\gamma}(2p-k)_{\delta)}\right). \tag{7}$$

The trace is worked out after extending the γ algebra to 2ω-dimensional space (Capper and Duff 1973), whereas the integral in (7) can be evaluated with the help of the integral formulae given in the appendix. After a straightforward but lengthy calculation the following expression is found for the pole part of the amplitude (7):

$$\begin{aligned}
T^{\text{pole}}_{\alpha\beta,\gamma\delta,\xi\eta}(k) = {} & \frac{2^2\kappa^3}{(4\pi)^2(2-\omega)32}\{k_\xi k_\eta(\tfrac{1}{30}E_{\alpha\beta,\gamma\delta} - \tfrac{1}{20}F_{\alpha\beta,\gamma\delta} - \tfrac{1}{15}k^2\delta_{\alpha\beta}\delta_{\gamma\delta} + \tfrac{1}{10}k^2 G_{\alpha\beta,\gamma\delta}) \\
& + \delta_{\xi\eta}(-\tfrac{1}{30}k_\alpha k_\beta k_\gamma k_\delta + \tfrac{1}{40}k^2 F_{\alpha\beta,\gamma\delta} - \tfrac{1}{60}k^2 E_{\alpha\beta,\gamma\delta} + \tfrac{1}{60}(k^2)^2\delta_{\alpha\beta}\delta_{\gamma\delta} - \tfrac{1}{40}(k^2)^2 G_{\alpha\beta,\gamma\delta}) \\
& + \tfrac{1}{120}(k_\xi H_{\alpha\beta\gamma\delta,\eta} + k_\eta H_{\alpha\beta\gamma\delta,\xi}) - \tfrac{1}{120}k^2(k_\alpha k_\beta G_{\gamma\delta,\xi\eta} + k_\gamma k_\delta G_{\alpha\beta,\xi\eta}) \\
& + \tfrac{1}{240}k^2(k_\alpha k_\gamma G_{\beta\delta,\xi\eta} + k_\alpha k_\delta G_{\beta\gamma,\xi\eta} + k_\beta k_\gamma G_{\alpha\delta,\xi\eta} + k_\beta k_\delta G_{\alpha\gamma,\xi\eta}) \\
& + \tfrac{1}{80}k^2[k_\xi(k_\alpha\delta_{\gamma\delta}\delta_{\beta\eta} + k_\beta\delta_{\gamma\delta}\delta_{\alpha\eta} + k_\gamma\delta_{\alpha\beta}\delta_{\delta\eta} + k_\delta\delta_{\alpha\beta}\delta_{\gamma\eta}) \\
& + k_\eta(k_\alpha\delta_{\gamma\delta}\delta_{\beta\xi} + k_\beta\delta_{\gamma\delta}\delta_{\alpha\xi} + k_\gamma\delta_{\alpha\beta}\delta_{\delta\xi} + k_\delta\delta_{\alpha\beta}\delta_{\gamma\xi})] \\
& - \tfrac{1}{160}k^2[k_\xi(k_\alpha G_{\gamma\delta,\beta\eta} + k_\beta G_{\gamma\delta,\alpha\eta} + k_\gamma G_{\alpha\beta,\delta\eta} + k_\delta G_{\alpha\beta,\gamma\eta}) \\
& + k_\eta(k_\alpha G_{\gamma\delta,\beta\xi} + k_\beta G_{\gamma\delta,\alpha\xi} + k_\gamma G_{\alpha\beta,\delta\xi} + k_\delta G_{\alpha\beta,\gamma\xi})] \\
& + \tfrac{1}{120}(k^2)^2(\delta_{\alpha\beta}G_{\gamma\delta,\xi\eta} + \delta_{\gamma\delta}G_{\alpha\beta,\xi\eta}) \\
& - \tfrac{1}{80}(k^2)^2(\delta_{\alpha\xi}G_{\gamma\delta,\beta\eta} + \delta_{\beta\xi}G_{\gamma\delta,\alpha\eta} + \delta_{\gamma\xi}G_{\alpha\beta,\delta\eta} + \delta_{\delta\xi}G_{\alpha\beta,\gamma\eta})\},
\end{aligned} \tag{8}$$

in which

$$E_{\alpha\beta,\gamma\delta} = k_\alpha k_\beta \delta_{\gamma\delta} + k_\gamma k_\delta \delta_{\alpha\beta}, \tag{9}$$

$$F_{\alpha\beta,\gamma\delta} = k_\alpha k_\gamma \delta_{\beta\delta} + k_\alpha k_\delta \delta_{\beta\gamma} + k_\beta k_\gamma \delta_{\alpha\delta} + k_\beta k_\delta \delta_{\alpha\gamma}, \tag{10}$$

$$G_{\alpha\beta,\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{11}$$

$$H_{\alpha\beta\gamma\delta,\xi} = \delta_{\alpha\xi}k_\beta k_\gamma k_\delta + \delta_{\beta\xi}k_\alpha k_\gamma k_\delta + \delta_{\gamma\xi}k_\alpha k_\beta k_\delta + \delta_{\delta\xi}k_\alpha k_\beta k_\gamma. \tag{12}$$

The fact that in equation (8) the tensorial indices (ξ, η) attached to the zero energy–momentum line do mix with the other indices makes the foregoing calculation non-trivial.

The diagram in figure 1 must also be completed by a similar diagram, differing from the former one by a change of loop direction. This is equivalent to an interchange of the (α, β) and (γ, δ) indices and an overall change of sign of the four-vector k. As equation (8) is invariant to these transformations, the total amplitude is given by twice the contribution expressed in equation (8).

3. Associated diagrams

To complete our proof, the two diagrams in figures 2 (*a*) and (*b*) must also be taken into consideration. A third diagram with the zero-momentum line on the right gives zero

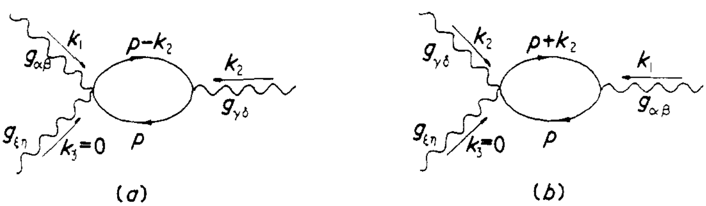

Figure 2. The two associated diagrams.

contribution in the context of dimensional regularization. (This would not be true in a theory with mass $\neq 0$).

The two-graviton two-fermion vertex with the same weight of the fermion field as in § 2 appears to be (see I):

$$
\begin{aligned}
V^4_{\alpha\beta,\gamma\delta}(k_1, k_2; p_1, p_2) \\
&= \tfrac{1}{16}\mathrm{i}(\gamma_{(\alpha}\delta_{\beta)\sigma}\delta_{\sigma(\gamma}q_{\delta)} + \gamma_{(\gamma}\delta_{\delta)\sigma}\delta_{\sigma(\alpha}q_{\beta)}) + \tfrac{1}{128}\mathrm{i}(\epsilon_{\alpha\gamma kn}\delta_{\beta\delta} + \epsilon_{\beta\gamma kn}\delta_{\alpha\delta} \\
&\quad + \epsilon_{\alpha\delta kn}\delta_{\beta\gamma} + \epsilon_{\beta\delta kn}\delta_{\alpha\gamma})(k_1 + k_2)_k\gamma_n\gamma_5 .
\end{aligned} \tag{13}
$$

It is easy to show that in 2ω-dimensional space the $\gamma_n\gamma_5$ part of equation (13) gives no contribution to the amplitude from the diagrams in figure 2. After a similar calculation to that in § 2 and also using the results of Capper *et al* (1973b) to evaluate the occurring integrals, the following expression is found for the pole part of the total amplitude of the diagrams of figure 2:

$$
\begin{aligned}
T^{\text{pole}}_{\alpha\beta,\gamma\delta,\xi\eta}(k) = \frac{2^2\kappa^3}{(4\pi)^2(2-\omega)64}\{ &-\tfrac{1}{30}(k_\xi H_{\alpha\beta\gamma\delta,\eta} + k_\eta H_{\alpha\beta\gamma\delta,\xi}) - \tfrac{1}{30}k^2(k_\alpha k_\beta G_{\gamma\delta,\xi\eta} + k_\gamma k_\delta G_{\alpha\beta,\xi\eta}) \\
&- \tfrac{1}{60}k^2[k_\xi(k_\alpha\delta_{\gamma\delta}\delta_{\beta\eta} + k_\beta\delta_{\gamma\delta}\delta_{\alpha\eta} + k_\gamma\delta_{\alpha\beta}\delta_{\delta\eta} + k_\delta\delta_{\alpha\beta}\delta_{\gamma\eta}) \\
&+ k_\eta(k_\alpha\delta_{\gamma\delta}\delta_{\beta\xi} + k_\beta\delta_{\gamma\delta}\delta_{\alpha\xi} + k_\gamma\delta_{\alpha\beta}\delta_{\delta\xi} + k_\delta\delta_{\alpha\beta}\delta_{\gamma\xi})] \\
&+ \tfrac{1}{20}k^2(k_\alpha k_\gamma G_{\beta\delta,\xi\eta} + k_\alpha k_\delta G_{\beta\gamma,\xi\eta} + k_\beta k_\gamma G_{\alpha\delta,\xi\eta} + k_\beta k_\delta G_{\alpha\gamma,\xi\eta}) \\
&+ \tfrac{1}{40}k^2[k_\xi(k_\alpha G_{\gamma\delta,\beta\eta} + k_\beta G_{\gamma\delta,\alpha\eta} + k_\gamma G_{\alpha\beta,\delta\eta} + k_\delta G_{\alpha\beta,\gamma\eta}) \\
&+ k_\eta(k_\alpha G_{\gamma\delta,\beta\xi} + k_\beta G_{\gamma\delta,\alpha\xi} + k_\gamma G_{\alpha\beta,\delta\xi} + k_\delta G_{\alpha\beta,\gamma\xi})] + \tfrac{1}{30}(k^2)^2(\delta_{\alpha\beta}G_{\gamma\delta,\xi\eta} + \delta_{\gamma\delta}G_{\alpha\beta,\xi\eta}) \\
&- \tfrac{1}{20}(k^2)^2(\delta_{\alpha\xi}G_{\gamma\delta,\beta\eta} + \delta_{\beta\xi}G_{\gamma\delta,\alpha\eta} + \delta_{\gamma\xi}G_{\alpha\beta,\delta\eta} + \delta_{\delta\xi}G_{\alpha\beta,\gamma\eta})\}.
\end{aligned} \tag{14}
$$

Adding twice the contribution of equation (8) to the contribution of equation (14), the pole part of the total third-order amplitude of our problem is found. This part will now be compared to the same order contribution originating from the counter-Lagrangian.

4. The counter-Lagrangian contribution

The counter term in the massless case is (see I):

$$
\Delta\mathscr{L} = -\frac{\sqrt{g}}{2-\omega}\frac{1}{(4\pi)^2}\frac{1}{60}(3R_{\mu\nu}R^{\mu\nu} - R^2). \tag{15}
$$

$R_{\mu\nu}$ is the Einstein tensor defined by:

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\rho,\nu} - \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} + \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho} \tag{16}$$

where

$$\Gamma^{\rho}_{\mu\nu} = \tfrac{1}{2}g^{\rho\sigma}(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}). \tag{17}$$

The counter term in equation (15) is four-dimensional and all further expansions proceed in four-dimensional Euclidean space. Using the well known parametrization:

$$\tilde{g}^{\mu\nu} = g^{\mu\nu}\sqrt{g} = \delta_{\mu\nu} + \kappa\phi_{\mu\nu}, \tag{18}$$

where κ is the gravitational coupling constant, the expressions for $(\sqrt{g})R^2$ and $(\sqrt{g})R_{\mu\nu}R^{\mu\nu}$ up to third order in κ can be evaluated. Therefore, $R_{\mu\nu}$ is expanded as follows:

$$R_{\mu\nu} = \kappa r_{\mu\nu} + \kappa^2 t_{\mu\nu} + O(\kappa^3), \tag{19}$$

where

$$r_{\mu\nu} = \tfrac{1}{2}[\phi_{\mu\rho,\nu\rho} + \phi_{\nu\rho,\mu\rho} - \phi_{\mu\nu,\rho\rho} + \tfrac{1}{2}\delta_{\mu\nu}\phi_{\lambda\lambda,\rho\rho}], \tag{20}$$

and

$$\begin{aligned} t_{\mu\nu} = \tfrac{1}{4}[&-\tfrac{1}{2}\phi_{\lambda\lambda,\mu}\phi_{\sigma\sigma,\nu} + \phi_{\rho\lambda,\mu}\phi_{\rho\lambda,\nu} + \phi_{\lambda\lambda,\rho}\phi_{\mu\nu,\rho} + 2\phi_{\mu\rho,\sigma}\phi_{\sigma\nu,\rho} - 2\phi_{\mu\rho,\lambda}\phi_{\nu\rho,\lambda} \\ &+ 2\partial_{\rho}(-\phi_{\mu\sigma}\phi_{\rho\sigma,\nu} - \phi_{\nu\sigma}\phi_{\rho\sigma,\mu} - \phi_{\rho\sigma}\phi_{\mu\nu,\sigma} + \phi_{\lambda\mu}\phi_{\lambda\nu,\rho} \\ &+ \phi_{\mu\lambda,\rho}\phi_{\nu\lambda} + \tfrac{1}{2}\delta_{\mu\nu}\phi_{\rho\sigma}\phi_{\lambda\lambda,\sigma} - \tfrac{1}{2}\delta_{\mu\nu}\phi_{\lambda\sigma}\phi_{\lambda\sigma,\rho} - \tfrac{1}{2}\phi_{\mu\nu}\phi_{\lambda\lambda,\rho})]. \end{aligned} \tag{21}$$

The third-order term of $(\sqrt{g})R^2$ is given by:

$$r_{\lambda\lambda}(2t_{\sigma\sigma} + 2\phi_{\mu\nu}r_{\mu\nu} - \tfrac{1}{2}\phi_{\sigma\sigma}r_{\rho\rho})\kappa^3, \tag{22}$$

and that of $(\sqrt{g})R_{\mu\nu}R^{\mu\nu}$ by:

$$r_{\mu\nu}(2t_{\mu\nu} + r_{\mu\sigma}\phi_{\sigma\nu} + r_{\sigma\nu}\phi_{\mu\sigma} - \tfrac{1}{2}\phi_{\lambda\lambda}r_{\mu\nu})\kappa^3. \tag{23}$$

As only the three-graviton diagram with one zero energy–momentum line is considered, only the terms that have no derivatives in at least one ϕ field are witheld. After this simplification, only terms with one ϕ field that is not differentiated appear. It is precisely on that field that the functional derivative $\delta/\delta\phi_{\xi\eta}$ operates, since the zero energy–momentum is associated with the (ξ, η) tensorial indices. Taking then subsequently the functional derivatives $\delta/\delta\phi_{\alpha\beta}$ and $\delta/\delta\phi_{\gamma\delta}$, going over to momentum space and carrying out the necessary symmetrization operations, the three-graviton diagram contributions from equations (22) and (23) are obtained. They read:

$$\begin{aligned} &T_{\alpha\beta,\gamma\delta,\xi\eta}(k)[(\sqrt{g})R] \\ &\quad = [k_{\xi}k_{\eta}(E_{\alpha\beta,\gamma\delta} + \delta_{\alpha\beta}\delta_{\gamma\delta}k^2) + \delta_{\xi\eta}(-k_{\alpha}k_{\beta}k_{\gamma}k_{\delta} - \tfrac{1}{2}k^2E_{\alpha\beta,\gamma\delta} - \tfrac{1}{4}(k^2)^2\delta_{\alpha\beta}\delta_{\gamma\delta}) \\ &\quad\quad - \tfrac{1}{2}k^2(k_{\alpha}k_{\beta}G_{\gamma\delta,\xi\eta} + k_{\gamma}k_{\delta}G_{\alpha\beta,\xi\eta}) - \tfrac{1}{4}(k^2)^2(\delta_{\alpha\beta}G_{\gamma\delta,\xi\eta} + \delta_{\gamma\delta}G_{\alpha\beta,\xi\eta})]\kappa^3, \end{aligned} \tag{24}$$

$$\begin{aligned} &T_{\alpha\beta,\gamma\delta,\xi\eta}(k)[(\sqrt{g})R_{\mu\nu}R^{\mu\nu}] \\ &\quad = \{k_{\xi}k_{\eta}(\tfrac{1}{2}E_{\alpha\beta,\gamma\delta} + \tfrac{1}{2}k^2G_{\alpha\beta,\gamma\delta} - \tfrac{1}{4}F_{\alpha\beta,\gamma\delta}) + \delta_{\xi\eta}[-\tfrac{1}{2}k_{\alpha}k_{\beta}k_{\gamma}k_{\delta} - \tfrac{1}{4}k^2E_{\alpha\beta,\gamma\delta} \\ &\quad\quad + \tfrac{1}{8}k^2F_{\alpha\beta,\gamma\delta} - \tfrac{1}{8}(k^2)^2G_{\alpha\beta,\gamma\delta}] - \tfrac{1}{4}k^2(k_{\alpha}k_{\beta}G_{\gamma\delta,\xi\eta} + k_{\gamma}k_{\delta}G_{\alpha\beta,\xi\eta}) \\ &\quad\quad + \tfrac{1}{8}k^2(k_{\alpha}k_{\gamma}G_{\beta\delta,\xi\eta} + k_{\alpha}k_{\delta}G_{\beta\gamma,\xi\eta} + k_{\beta}k_{\gamma}G_{\alpha\delta,\xi\eta} + k_{\beta}k_{\delta}G_{\alpha\gamma,\xi\eta}) \\ &\quad\quad - \tfrac{1}{8}(k^2)^2(\delta_{\alpha\xi}G_{\gamma\delta,\beta\eta} + \delta_{\beta\xi}G_{\gamma\delta,\alpha\eta} + \delta_{\gamma\xi}G_{\alpha\beta,\delta\eta} + \delta_{\delta\xi}G_{\alpha\beta,\gamma\eta})\}\kappa^3. \end{aligned} \tag{25}$$

The three-graviton amplitude for the counter term of equation (15) thus becomes:

$$
\begin{aligned}
T_{\alpha\beta,\gamma\delta,\xi\eta}(k)&\left(-\frac{\sqrt{g}}{(2-\omega)}\frac{1}{(4\pi)^2}\frac{1}{60}(3R_{\mu\nu}R^{\mu\nu}-R^2)\right)\\
&=\frac{\kappa^3}{(4\pi)^2(2-\omega)60}\{k_\xi k_\eta(-\tfrac{1}{2}E_{\alpha\beta,\gamma\delta}+k^2\delta_{\alpha\beta,}\delta_{\gamma\delta}-\tfrac{3}{2}k^2G_{\alpha\beta,\gamma\delta}+\tfrac{3}{4}F_{\alpha\beta,\gamma\delta})\\
&\quad+\delta_{\xi\eta}[\tfrac{1}{2}k_\alpha k_\beta k_\gamma k_\delta+\tfrac{1}{4}k^2E_{\alpha\beta,\gamma\delta}-\tfrac{1}{4}(k^2)^2\delta_{\alpha\beta}\delta_{\gamma\delta}-\tfrac{3}{8}k^2F_{\alpha\beta,\gamma\delta}+\tfrac{3}{8}(k^2)^2G_{\alpha\beta,\gamma\delta}]\\
&\quad+\tfrac{1}{4}k^2(k_\alpha k_\beta G_{\gamma\delta,\xi\eta}+k_\gamma k_\delta G_{\alpha\beta,\xi\eta})\\
&\quad-\tfrac{3}{8}k^2(k_\alpha k_\gamma G_{\beta\delta,\xi\eta}+k_\alpha k_\delta G_{\beta\gamma,\xi\eta}+k_\beta k_\gamma G_{\alpha\delta,\xi\eta}+k_\beta k_\delta G_{\alpha\gamma,\xi\eta})\\
&\quad-\tfrac{1}{4}(k^2)^2(\delta_{\alpha\beta}G_{\gamma\delta,\xi\eta}+\delta_{\gamma\delta}G_{\alpha\beta,\xi\eta})\}.
\end{aligned}
\tag{26}
$$

This is exactly the expression found by adding twice equation (8) to equation (14), as may be verified directly. This completes our proof.

Considering equations (25) and (26), it is seen that the terms which contain the factor $\delta_{\xi\eta}$ are second-order amplitudes, satisfying the second-order Ward identities quoted in I. Calculating the trace with respect to the indices ξ and η in both equations yields the second-order contributions (in κ) of the corresponding parts of the counter term apart from a numerical factor -2.

5. Summary

It has been shown that the self-energy counter term Lagrangian cancels the infinities of the fermion triangle loop with three graviton lines attached, in the particular case that one of the gravitons has zero energy–momentum. The technique and some results outlined in the present paper could be of use in proving the same statement in the case of a scalar particle loop, of a photon loop or of a neutrino loop, although the calculations may become more tedious.

Acknowledgments

We express our gratitude to Dr D M Capper for suggesting the problem, and to Professor Dr C C Grosjean and Dr J Krüger for a critical reading of the manuscript.

Appendix

In this appendix, a method is given to calculate integrals of the type:

$$
J=\int\frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}}\frac{f(p_\alpha)}{(p^2)^2(p-k)^2},
\tag{A.1}
$$

where $f(p_\alpha)$ represents a polynomial in p_α of degree not greater than six. If $f(p_\alpha)$ can be factorized such that p^2 appears as a factor, we are immediately led to the integrals calculated by Capper *et al* (1973b). To see how the method works in general, the integral

J_1 is calculated in detail:

$$J_1 = \int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} \frac{1}{(p^2)^2(p-k)^2}. \tag{A.2}$$

Using three times the parametrization:

$$\frac{1}{q^2} = \int_0^\infty \mathrm{d}\alpha\, \mathrm{e}^{-\alpha q^2}, \tag{A.3}$$

together with the formula

$$\int \mathrm{d}^{2\omega}p \exp(-ap^2 \pm 2bp) = \left(\frac{\pi}{a}\right)^\omega \mathrm{e}^{b^2/a} \qquad \mathrm{Re}(a) > 0. \tag{A.4}$$

J_1 can be reduced to the form:

$$J_1 = \frac{1}{(4\pi)^\omega} \int_0^\infty \mathrm{d}\alpha \int_0^\infty \mathrm{d}\beta \int_0^\infty \mathrm{d}\gamma (\alpha+\beta+\gamma)^{-\omega} \exp\left(-\frac{\alpha(\beta+\gamma)}{\alpha+\beta+\gamma}k^2\right). \tag{A.5}$$

Making the substitutions $\gamma = \beta t$, $\alpha = (1+t)\beta s$, one finds:

$$J_1 = \frac{1}{(4\pi)^\omega} \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}\beta \beta^{2-\omega}(1+t)^{1-\omega}(1+s)^{-\omega} \exp\left(-\beta\frac{s(1+t)}{(1+s)}k^2\right). \tag{A.6}$$

Integration with respect to β and t yields:

$$\begin{aligned} J_1 &= \frac{1}{(4\pi)^\omega}\Gamma(3-\omega)(k^2)^{\omega-3} \int_0^\infty \mathrm{d}s(1+s)^{3-2\omega}s^{\omega-3} \\ &= \frac{1}{(4\pi)^\omega}\Gamma(3-\omega)(k^2)^{\omega-3}B(\omega-2, \omega-1), \end{aligned} \tag{A.7}$$

where $B(x, y)$ is Euler's integral of the first kind (Gradshteyn and Ryzhik 1965). The other integrals J_j corresponding to polynomials of degree $j-1$ can be evaluated in a similar manner, using differentiation under the integral sign in (A.4). This leads to:

$$J_1 = \int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} \frac{1}{(p^2)^2(p-k)^2} = I_1, \tag{A.8}$$

$$J_2 = \int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} \frac{p_\alpha}{(p^2)^2(p-k)^2} = k_\alpha I_2, \tag{A.9}$$

$$J_3 = \int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} \frac{p_\alpha p_\beta}{(p^2)^2(p-k)^2} = k_\alpha k_\beta I_3 + \delta_{\alpha\beta} I_4, \tag{A.10}$$

$$J_4 = \int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} \frac{p_\alpha p_\beta p_\gamma}{(p^2)^2(p-k)^2} = k_\alpha k_\beta k_\gamma I_5 + \sum_{3p} k_\alpha \delta_{\beta\gamma} I_6, \tag{A.11}$$

$$J_5 = \int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} \frac{p_\alpha p_\beta p_\gamma p_\delta}{(p^2)^2(p-k)^2} = k_\alpha k_\beta k_\gamma k_\delta I_7 + \sum_{6p} k_\alpha k_\beta \delta_{\gamma\delta} I_8 + \sum_{3p} \delta_{\alpha\beta}\delta_{\gamma\delta} I_9, \tag{A.12}$$

$$J_6 = \int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} \frac{p_\alpha p_\beta p_\gamma p_\delta p_\lambda}{(p^2)^2(p-k)^2} = k_\alpha k_\beta k_\gamma k_\delta k_\lambda I_{10} + \sum_{10p} k_\alpha k_\beta k_\gamma \delta_{\delta\lambda} I_{11} + \sum_{15p} k_\alpha \delta_{\beta\gamma}\delta_{\delta\lambda} I_{12}, \tag{A.13}$$

$$J_7 = \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{p_\alpha p_\beta p_\gamma p_\delta p_\lambda p_\rho}{(p^2)^2(p-k)^2} = k_\alpha k_\beta k_\gamma k_\delta k_\lambda k_\rho I_{13}$$

$$+ \sum_{15p} k_\alpha k_\beta k_\gamma k_\delta \delta_{\lambda\rho} I_{14} + \sum_{45p} k_\alpha k_\beta \delta_{\gamma\delta}\delta_{\lambda\rho} I_{15} + \sum_{15p} \delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\lambda\rho} I_{16}, \tag{A.14}$$

where the summations are restricted to only those permutations of the indices which give rise to essentially different terms. The functions I are given by:

$$I_1 = \frac{1}{(4\pi)^\omega}\Gamma(3-\omega)(k^2)^{\omega-3}B(\omega-2, \omega-1), \tag{A.15}$$

$$I_2 = \frac{1}{(4\pi)^\omega}\Gamma(3-\omega)(k^2)^{\omega-3}B(\omega-1, \omega-1), \tag{A.16}$$

$$I_3 = \frac{1}{(4\pi)^\omega}\Gamma(3-\omega)(k^2)^{\omega-3}B(\omega, \omega-1), \tag{A.17}$$

$$I_4 = \frac{1}{(4\pi)^\omega}\Gamma(2-\omega)(k^2)^{\omega-2}B(\omega-1, \omega)\tfrac{1}{2}, \tag{A.18}$$

$$I_5 = \frac{1}{(4\pi)^\omega}\Gamma(3-\omega)(k^2)^{\omega-3}B(\omega+1, \omega-1), \tag{A.19}$$

$$I_6 = \frac{1}{(4\pi)^\omega}\Gamma(2-\omega)(k^2)^{\omega-2}B(\omega, \omega)\tfrac{1}{2}, \tag{A.20}$$

$$I_7 = \frac{1}{(4\pi)^\omega}\Gamma(3-\omega)(k^2)^{\omega-3}B(\omega+2, \omega-1), \tag{A.21}$$

$$I_8 = \frac{1}{(4\pi)^\omega}\Gamma(2-\omega)(k^2)^{\omega-2}B(\omega+1, \omega)\tfrac{1}{2}, \tag{A.22}$$

$$I_9 = \frac{1}{(4\pi)^\omega}\Gamma(1-\omega)(k^2)^{\omega-1}B(\omega, \omega+1)\tfrac{1}{4}, \tag{A.23}$$

$$I_{10} = \frac{1}{(4\pi)^\omega}\Gamma(3-\omega)(k^2)^{\omega-3}B(\omega+3, \omega-1), \tag{A.24}$$

$$I_{11} = \frac{1}{(4\pi)^\omega}\Gamma(2-\omega)(k^2)^{\omega-2}B(\omega+2, \omega)\tfrac{1}{2}, \tag{A.25}$$

$$I_{12} = \frac{1}{(4\pi)^\omega}\Gamma(1-\omega)(k^2)^{\omega-1}B(\omega+1, \omega+1)\tfrac{1}{4}, \tag{A.26}$$

$$I_{13} = \frac{1}{(4\pi)^\omega}\Gamma(3-\omega)(k^2)^{\omega-3}B(\omega+4, \omega-1), \tag{A.27}$$

$$I_{14} = \frac{1}{(4\pi)^\omega}\Gamma(2-\omega)(k^2)^{\omega-2}B(\omega+3, \omega)\tfrac{1}{2}, \tag{A.28}$$

$$I_{15} = \frac{1}{(4\pi)^\omega}\Gamma(1-\omega)(k^2)^{\omega-1}B(\omega+2, \omega+1)\tfrac{1}{4}, \tag{A.29}$$

$$I_{16} = \frac{1}{(4\pi)^{\omega}}\Gamma(-\omega)(k^2)^{\omega}B(\omega+1, \omega+2)\tfrac{1}{8}, \tag{A.30}$$

For our purposes, it has been sufficient to know the pole term in the Laurent expansion around $\omega = 2$ of the expressions (A.15) to (A.30).

Finally, we wish to point out an interesting simplification which is possible in the case of integrals of the type:

$$K = \int d^{2\omega}p \frac{2pkf(p_\alpha)}{(p^2)^2(p-k)^2}. \tag{A.31}$$

It is based on the identity:

$$2pk = p^2 + k^2 - (p-k)^2. \tag{A.32}$$

Indeed, substituting equation (A.31), we obtain a sum of three integrals, the first one being of a simpler type, the second one having a numerator of lower degree in p_α its integrand and the third one being zero in the dimensional regularization scheme.

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